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The confining behavior and asymptotic freedom for QCD($SU(\infty)$) – a constant-gauge-field path-integral analysis

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Abstract. We examine the Wilson confining-area behavior for QCD($SU(\infty)$) as described by a Witten effective reduced dynamics of constant-gauge fields. We show the model to be exactly soluble when dynamical quark fields are added, together with fermion asymptotic freedom behavior. Additionally, we give arguments for the triviality of the chiral-$SU(N)$ non-abelian Thirring model at the t´Hooft limit of large number of colors $N \to \infty$ in the context of these constant-gauge-field reduced dynamics.

1 Introduction

Since 1950, the quantum field theory of light and electrons (QED) has been a highly consistent framework for the description of the interaction of light and charged matter. In 1967, this quantum field theory of particles achieved another success with the advent of the Weinberg-Salam quantum field theory, which successfully handled the weak-electromagnetic component of the nuclear scattering processes.

These quantum field methods are based on a principle of minimal action with (local and global) symmetries and the existence of a mathematical generating functional (Schwinger) defined on the space of classical source fields (test functions in the language of Schwart distribution theory). This generating functional in turn contains all the probabilities for occurrences associated with all the physically possible quantum scatterings involving the elementary particle-field excitations.

However, until now it has been a difficult challenge to apply these scattering quantum-field (LSZ) methods directly to the description of pure strong-nuclear interaction such as a particle-field theory based on the framework of the non-abelian gauge theory of quantum chromodynamics (QCD). The basic and conceptual difficulty in applying the LSZ quantum-field method to quantum chromodynamics is rooted in the first assumption of the QCD model: charge-color confinement; all QCD particles must be subject to this, which in turn constrains particles with only a color-singlet compound structure to be subject to physical LSZ scattering processes.

It is important to remark that strong mathematical clues for this QCD charge-color confinement were obtained by Wilson (1974) in a discretized spacetime by using as dynamical variables the well-known gauge-invariant discretized Mandelstam-Feynman phase factors instead of the gauge-variant discretized fields. Although there is a strong indication that it is possible to remove the difficulties of the direct use of a discrete spacetime through a second-order phase transition leading to a zero lattice-spacing limit, this step remains a somewhat unsolved problem within Wilson's program for QCD to the present day.

The purpose of this paper is to consider another quantum Yang-Mills reduced model with an explicitly confining behavior at the limit of large number of charge-colours (the t'Hooft limit), but which is defined on a continuum spacetime. This reduced quantum-dynamical model is defined by introducing directly on R^{ν}, a functional manifold of constant-gauge-field configurations [1], which in turn are expected to generate an effective dynamics on the manifold of the full gauge-field configurations at the t'Hooft limit $SU(\infty)$ for the Yang-Mills path integral. We show the Wilson confining-area behavior for QCD($SU(\infty)$) as described by our proposed $SU(\infty)$ effective reduced dynamics of constant-gauge fields. We show the exact solubility of our $SU(\infty)$ model with the addition of full dynamical quark fields and the related fermionic field asymptotic freedom. These studies are presented in Sect. 3 of this paper.

Another interesting and conceptually important problem in quantum field theory is to understand the triviality of quantum field theories as a phase-transition phenomena depending on external parameters, including the famous spacetime dimensionality.

It is argued sometimes that there are no non-renormalizable quantum field theories. What is really happening is the appearance of the quantum field theory triviality phenomena. However, there is some analysis in the literature pointing out that through resummations – especially by means of the large-N expansions – one could make such non-renormalizable Field theories (such as the Thirring fermion quantum field model) turn out to be nontrivial renormaliz-

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able ones. We aim in Sect. 3 to present an analysis, based on an approximate chiral path-integral bosonization and the Witten reduced constant-gauge-field dynamics of Sect. 2, to show that such resummation renormalization phenomenon does not happen. In Sect. 4 we complement our previous path-integral analysis by presenting a triviality argument by means of a loop space analysis for any N.

2 The model and its confining behavior

One of the basic quantum field variables used to probe the nonperturbative phase of non-abelian gauge field theories is the well-known (Euclidean) path-integral average associated with the non-abelian Faraday flux defined by a space-time loop C: the so-called Wilson-Mandelstam loop variable

$$W[C] = \frac{1}{W(0)}\left\{\int_{S'(R^\nu\times SU(N))} D^F[A_\mu(x)] \times \exp\left(-\frac{1}{2}\int_{R^\nu}\mathrm{Tr}(F_{\mu\nu})^2(x)d^\nu x\right) \times \left(\frac{1}{N}\,\mathrm{Tr}\,\mathbb{P}\left[\exp\left(ig\oint_C A_\mu\,dx_\mu\right)\right]\right)\right\} \quad (1)$$

where the domain of the quantum average of (1) is composed of Schwartz-tempered $SU(N)$-valued connections associated with the bundle $R^\nu\times SU(N)$.

A long time ago [1], it was argued by E. Witten that. at the limit of an infinite number of colors $N\to\infty$ with the diagrammatic restriction $\lim_{N\to\infty}(g^2N)=g^2_\infty<\infty$, the full domain of the Yang-Mills functional integral (1) would be expected to be reduced to a manifold of translation-invariant constant-gauge fields. Let us, thus, define our reduced Yang-Mills model by considering from the beginning only constant-gauge-field configurations on the functional domain of (1) as our basic assumption.

We now show the usefulness of such effective dynamics by giving a proof of the color-charge confinement through an explicit evaluation of the Wilson-Mandelstam phase factor at $N\to\infty$, an important result supporting the possibility of the above reduction of degrees of freedom for Yang-Mills theory at $SU(\infty)$, as first conjectured in [1].

The main idea behind making this path-integral evaluation for constant-gauge fields explicit is to consider the (non-gauge-invariant) Cartan decomposition of each constant gauge field A_μ in the path-integral average (1).

$$A_\mu = B^a_\mu\,H_a + G^b_\mu\,E_b \quad (2)$$

where the Cartan basis $\{H_a, E_a\}$ of the $SU(N)$ Lie algebra have the following distinguished calculational properties [2]

a) For $a,b=1,2,\ldots,N-1$

$$[H_a,H_b]_- = 0\,. \quad (3)$$

b) For $b=\pm1,\ldots,\pm\dfrac{N(N-1)}{2}$

$$[H_a,E_b]_- = r_a(b)E_b\,. \quad (4)$$

c) For $a=1,2,\ldots,\dfrac{N(N-1)}{2}$

$$[E_a,E_{-a}]_- = \sum_{\ell=1}^{N-1} r_c(a)H_a\,. \quad (5)$$

d) For $a\neq -b;\quad a,b=\pm1,\ldots,\pm\dfrac{N(N-1)}{2}$

$$[E_a,E_b]_- = N_{ab}\,E_{a+b}\,. \quad (6)$$

Since one has to fix the gauge on the path-integral (1) and at the same time one should preserve the non-abelian nature of the field variable, which is expected to be dynamically significant to explain charge confinement, we impose that the abelian components should vanish as our gauge-fixing condition (the Bollini-Giambiagi gauge, see the last reference of [1]).

$$B^a_\mu \equiv 0\,. \quad (7)$$

Note that the use of the gauge-fixing condition allows us to simplify considerably the objects to be path-integrated on our proposed $SU(\infty)$ constant-gauge-field model.

For instance, the constant-gauge-field Yang-Mills path-integral weight is obtained by simply substituting (2) into the Yang-Mills action, which leads to a pure fourth-order polynomial action

$$\begin{aligned} S[G^b_\mu\,E_b] &= \frac{1}{2}\int_\Omega d^\nu x\left(\mathrm{Tr}(\partial_\mu A_\nu-\partial_\nu A_\mu+ig[A_\mu,A_\nu])^2\right) \\ &= -\frac{g^2}{2}\cdot V\;\mathrm{Tr}([G_\mu,G_\nu]^2) \\ &= -\frac{g^2}{2}\,V\,G^a_\mu\,G^b_\nu\,G^c_\mu\,G^d_\nu[\mathcal{L}_{abcd}]\,. \end{aligned} \quad (8)$$

Here we have introduced an appropriate finite-volume domain $\Omega\subset R^\nu$ such that $\mathrm{vol}(\Omega)=V$ and with the topology product form $\Omega=S\times[0,\ell_3]\times[0,\ell_4]$ to extract the area behavior of (1) at the limit of large-area behavior $S\to\infty$ (infinite volume V). The matrix of color indices $\mathcal{L}_{abcd}$ is given explicitly by (with $\mathrm{Tr}(E_aE_b)=+2\delta_{ab}$)

$$\mathcal{L}_{abcd} = \left(\sum_{i,\ell=1}^{N-1} r_i(a)r_\ell(c)\delta_{i\ell}\,\delta_{c,-d}\,\delta_{a,-b}\right) + \left(N_{ab}\,N_{cd}(1-\delta_{a,-b})(1-\delta_{c,-d})\delta_{a+b,-(c+d)}\right)\,. \quad (9)$$

We have the following exact result for the Mandelstam phase factor as a straightforward consequence of the non-abelian Stokes theorem applied to the planar loop C, which is supposed to be entirely contained in the plane ($\mu=0$, $\nu=1$, (containing the Euclidean time axis) and S denotes the area of the minimal surface bounded by C with the disc topology (for a rigorous proof see Sect. 3).

$$\mathbb{P}\left\{e^{ig\oint_{C_{0,1}}A_\mu\,dx_\mu}\right\} = \exp\left(-g^2S\;\mathrm{Tr}[A_0,A_1]\right)\,. \quad (10)$$

The leading limit of $N\to\infty$ in (10) (similarly to the deduction of the large-number law in statistics) yields the closed result below

$$\frac{1}{N}\ \mathrm{Tr}\ \mathbb{P}\left\{e^{iS\oint_{C_{0,1}} A_\mu\, dx_\mu}\right\}$$
$$= \exp\left\{+\frac{(g^2S)^2}{2N}\ (\mathrm{Tr}[A_0,A_1])^2\right\} + O\left(\frac{1}{N}\right)$$
$$= \exp\left\{+\frac{(g^2S)^2}{2N}\ G^a_\mu\, G^b_\nu\, G^c_\mu\, G^d_\nu [\mathcal{L}_{abcd}]\delta_{\mu 0}\,\delta_{\nu 1}\right\}\ . \quad (11)$$

At this point of our path-integral study, let us make a technical remark not used in what follows and related to the fact that the path-integral average (1) for constant-gauge fields is fully $SU(N)$ gauge-invariant and, as a consequence, one should in principle evaluate the Faddev-Papov Jacobian associated with our proposed gauge-fixing (7). To implement this technical step, one considers the infinitesimal functional displacements through a gauge transformation with parameters $[\delta\omega^a, \delta\varepsilon^b\}$

$$\delta A_\mu = \left\{\left(\delta G^b_\mu\right) E_b + i\left(\delta\omega^a\right)\left(G^{b'}_\mu\, E_{b'}\right)\left(-r_a\left(b'\right)\right)\right.$$
$$+\, i\left(\delta\varepsilon^b\right)\left[G^{b'}_\mu\,\delta_{b,-b'}\left(\sum_{\ell=1}^{N-1} r_\ell(b) H_\ell\right)\right]$$
$$\left. +\, i\left(\delta\varepsilon^b\right)\left[G^{b'}_\mu\, N_{bb'}\, E_{b+b'}\left(1-\delta_{b,-b'}\right)\right]\right\}, \quad (12)$$

which after substituting in the functional metric [3],

$$\delta s^2_A = \mathrm{Tr}\left(\int_\Omega (\delta A\cdot\delta A) d^\nu x\right)$$
$$= [\delta\sigma,\delta\varepsilon,\delta\omega]^T\, M[\sigma,\varepsilon,\omega][\delta\sigma,\delta\varepsilon,\delta\omega]\,, \quad (13)$$

would lead us to the Faddev-Popov Jacobian as the functional metric determinant averaged over the gauge group (with the infinitesimal gauge group neighborhood implying the use of the Feynman measure)

$$\Delta_{\mathrm{F}P}[G_\mu] = \int_{SU(N)} D^F(\delta\varepsilon,\delta\omega)\ \det{}^{\frac{1}{2}}\left\{M[\bar\sigma,\delta\varepsilon,\delta\omega]\right\}\ . \quad (14)$$

However, it is expected that in the large N limit (14) does not affect the confining area behavior of the averaged Wilson loop equation (1). We thus neglect its contribution to the average equation (1).

$$\Delta_{\mathrm{F}P}[G_\mu] = 1 + O\left(\frac{1}{N}\right)\ . \quad (15)$$

By collecting (8) and (11), one finally obtains our proposed path-integral representation for the Wilson loop for constant-gauge fields for large number of colors $N\to\ \infty$.

$$W[C_{01}]$$
$$= \lim_{N\to\infty}\left\{\frac{1}{W(0)}\int\left(\prod_{a=1}^{N^2-N}\prod_{\mu=0}^{\nu-1} dG^a_\mu\right)\right\}$$
$$\times\exp\left\{+\frac{1}{2}\, G^a_\mu\, G^b_\nu\, G^c_\mu\, G^d_\nu\,\mathcal{L}_{abcd}\right.$$
$$\left.\times\left[g^2V + \delta_{\mu 0}\delta_{\nu 1}\frac{\left(g^2S\right)^2}{N}\right]\right\}. \quad (16)$$

Now the area behavior at the t'Hooft limit of large number of colors $N\to\infty$ is obtained exactly after considering a simple rescaling of the G^a_μ variables in both path-integral factors in (16) (including the normalization factor $W(0)$), namely $G^a_{(0,1)}\to G^a_{(0,1)}\left[g^2V+\dfrac{(g^2S)^2}{N}\right]^{-\frac{1}{4}}$ in the numerator and $G^a_\mu\to G^a_\mu[g^2V]^{-\frac{1}{4}}$ in the denominator as well.

$$W[C] = \frac{\left[g^2V\left(1+\frac{g^2S^2)}{NV}\right)\right]^{-\frac{(N^2-N)\nu}{4}}}{[g^2V]^{-(N^2-N)\frac{\nu}{4}}}$$
$$= \left(1+\frac{g^2S^2}{NV}\right)^{-\frac{N(N-1)\nu}{4}} \quad (17)$$

which, in the large-N limit, gives us exactly the expected exponential area behavior in a four-dimensional space time of the cylindrical form $\Omega^{(\infty)} = R^2\times[0,\ell_3]\times[0,\ell_4]$, with $S\to\infty$ (the area bounded by C).

$$W[C]\sim\exp_{S\to\infty}\left\{-\frac{\left(\lim_{N\to\infty}(g^2(N-1))\right)}{(\ell_3\ell_4)S}\cdot S^2\right\}$$
$$\sim\exp\left\{-\left(\frac{g^2_\infty}{(\ell_3\ell_4)}\right)S\right\}\ . \quad (18)$$

It is very important to point out the appearance of a kind of dual models-string slope parameter $\dfrac{g^2_\infty}{(\ell_3\ell_4)}$ as an overall coefficient in the area behavior equation (18), which signals the existence of the phenomenon of dimensional transmutation on the adimensional $SU(\infty)$ gauge-coupling constant in four-dimensional spacetime, a phenomena expected to be responsible for the existence of strings structures on QCD($SU(\infty)$)as well as generating the expected mass scale for hadrons in the observed nuclear particle forces [4]. Note that the string tension on (18) depends solely on the area vacuum cross section $A=\ell_3\ell_4$, as expected [4]. In the three-dimensional case one obtains a pure length behavior for the Wilson loop on the basis of (18).

Finally, in the two-dimensional case one obtains the area behavior, however without the phenomenon of dimensional transmutation for the $N=\infty$ coupling constant [4].

After producing arguments for the confining behavior in our reduced constant-gauge-field model through explicit evaluation, we now introduce full dynamical chiral fermion fields in our proposed constant-gauge-field Yang-Mills $SU(\infty)$ theory.

The associated quark-field-generating functional in the presence of the background constant-gauge fields can be explicitly evaluated.

Let us briefly show this result since we make a complete analysis of this problem in the next Sect. 3. Firstly we have the following chiral quark-field Euclidean path integral

$$Z[\eta,\bar\eta]=\frac{1}{Z(0,0)}\int D^F[\psi(x)]D^F[\bar\psi(x)]\delta^{(F)}(\gamma_5\psi-\psi)\times\delta^{(F)}(\gamma_5\bar\psi-\bar\psi)\times\exp\Bigg\{-\frac{1}{2}\int_\Omega d^\nu x(\psi,\bar\psi)\times\begin{bmatrix}\bigcirc & U(\phi)\,\partial\!\!\!/ U(\phi)\\ U(\phi)^*\;\partial\!\!\!/^* U^*(\phi) & \bigcirc\end{bmatrix}\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}\Bigg\}\times\exp\left\{-i\int_\Omega(\bar\psi\eta+\bar\eta\psi)d^\nu x\right\} \tag{19}$$

where the chiral $SU(N)$ phase $U(\phi)$ associated with the constant-gauge fields configuration is given explicitly by the expression

$$U(\phi)=\{\exp\left[-ig\,\gamma_5(A^a_\alpha\cdot x^\alpha)\lambda_a\right]\}=\mathbb{P}\left\{e^{-ig\gamma_5\int_{-\infty}^x A^a_\alpha\cdot dx^\alpha}\right\} \tag{20}$$

where $\phi=\phi^a\lambda_a=A^a_\alpha\,x^\alpha\lambda_a$ is the chiral phase.

We can proceed as in the chiral bosonization path integral framework in order to "bosonize" (solve exactly) the quark-field path-integral (19) by means of the chiral change of variables [5]

$$\psi(x)=\exp\{-ig\,\gamma_5\,\phi(x)\}\chi(x)\,,\qquad \bar\psi(x)=\overline{\chi}(x)\,\exp\{-ig\,\gamma_5\,\phi(x)\}\,. \tag{21}$$

After the change equation (21), the generating functional takes the decoupled form

$$Z[\eta,\bar\eta]=\frac{1}{Z(0,0)}\int D^F[\chi(x)]D^F[\bar\chi(x)]\times\exp\left\{-\frac{1}{2}\int_\Omega d^\nu x(\chi,\bar\chi)(x)\begin{bmatrix}\bigcirc & \partial\!\!\!/\\ \partial\!\!\!/^* & \bigcirc\end{bmatrix}\begin{pmatrix}\chi\\ \bar\chi\end{pmatrix}(x)\right\}\times\exp\Bigg\{-\frac{i}{2}\int_\Omega d^\nu x\times\left(\overline{\chi}\,e^{-ig\gamma_5\,\phi(x)}\eta+\bar\eta\,e^{-ig\,\gamma_5\,\phi(x)}\chi\right)(x)\Bigg\}\times\det{}^{+1}_F[U(\phi)\;\partial\!\!\!/ U(\phi)]\,. \tag{22}$$

At this point, we remark the validity of the free-field result for the fermionic functional determinant in the path-integrand equation (22) (see the next section for detailed calculations)

$$\det{}_F[U(\phi)\;\partial\!\!\!/ U(\phi)]=\det{}_F\,[\partial\!\!\!/]\,. \tag{23}$$

Here we have used the Alvarez-Romanov-Schwartz theorem [5], the condition $\int_\Omega d^\nu x\cdot x^\mu=0$ and the non-existence of zero modes of the Dirac operator in presence of constant gauge field configurations in order to obtain (23).

As a consequence of the above displayed results, one gets the famous asymptotic freedom property of the quark fields in our $SU(\infty)$ constant gauge field model after writing explicitly the quark two-point function

$$\langle\psi(x)\bar\psi(y)\rangle=\frac{\delta^2 Z[\eta,\bar\eta]}{\delta\bar\eta(x)\delta\eta(y)}\bigg|_{\eta=\bar\eta=0}=\langle\chi(x)\overline{\chi}(y)\rangle^{(0)}\exp\left(-ig\,\gamma_5\int_x^y A_\mu\,dx_\mu\right)\underset{|x-y|\to 0}{\sim}\langle\chi(x)\overline{\chi}(y)\rangle^{(0)}\,. \tag{24}$$

Here $\langle\chi(x)\overline{\chi}(y)\rangle$ denotes the free fermion propagator coming from the "bosonized" action and the contour on the gauge field path-phase factor is a straight line connecting the points x^α and y^α, which reduces to unity at the higher-energy limit of $|x-y|\to 0$ [see (20)].

At this point, let us call the reader's attention to the fact that phenomenon of asymptotic freedom should be analyzed for gauge-invariant quark bilinear fields. For instance, we have the gauge-invariant result:

$$\langle(\psi(x)\bar\psi(x))(\psi(y)\bar\psi(y))\rangle\sim\langle\chi(x)\bar\chi(y)\rangle^{(0)}\langle\chi(y)\bar\chi(x)\rangle^{(0)}\times\left\{\mathrm{Tr}_{SU(\infty)}\,P(+ig\oint_{C_{xy}}A_\mu dx_\mu)\right\}\,. \tag{24b}$$

Here C_{xy} denotes an arbitrary planar closed contour intercepting the marked points x and y. We can see that, for large $|x-y|$ separation, the above quark-bilinear field correlation function approximates the free-field fermion correlation functions as the family of planar loops C_{xy} in the gauge-invariant expression (24)b reduces to a point as the geometrical result of the superposition of the segments of the straight line connecting the points x and y (see (24)), however with opposite orientation. Note that all those loops C_{xy} with a large area $|x-y|^2$ make a negligible contribution to (24)b.

3 The path-integral triviality argument for the Thirring model at $SU(\infty)$

We start our analysis by considering the chiral non-abelian $SU(N_c)$ Thirring model Lagrangian on the Euclidean space-time of finite volume $\Omega\subset R^4$, as in Sect. 2

$$L(\psi,\overline{\psi})=\frac{1}{2}\left[\overline{\psi}^a\left(i\overrightarrow{\gamma_\mu\partial_\mu}\psi^a\right)+\left(\overline{\psi}^a i\overleftarrow{\gamma_\mu\partial_\mu}\right)\psi^a\right]+\left(\frac{g^2}{2}\left(\overline{\psi}_b\gamma^\mu\gamma^5\left(\lambda^A\right)_{bc}\psi_c\right)^2\right)\,. \tag{25}$$

Here $(\psi^a,\overline{\psi}^a)$ are the Euclidean four-dimensional chiral fermion fields belonging to a fermionic fundamental representation of the $SU(N_c)$ non-abelian group with Dirichlet boundary condition imposed at the finite-volume region Ω. In the framework of path integrals, the generating functional of the Green's functions of the quantum field theory associated with the Lagrangian (25) is given by ($\partial\!\!\!/ = i\,\gamma_\mu \partial_\mu$)

$$Z[\eta_a, \overline{\eta}_a] = \frac{1}{Z(0,0)} \int \prod_{a=1}^{N^2-N} D[\psi_a] D[\overline{\psi}_a] \times \exp\left\{ -\frac{1}{2} \int_\Omega \mathrm{d}^4x\, (\psi_a, \overline{\psi}_a) \begin{bmatrix} 0 & \partial\!\!\!/ \\ \overleftarrow{\partial\!\!\!/}^* & 0 \end{bmatrix} \begin{pmatrix} \psi_a \\ \overline{\psi}_a \end{pmatrix} (x) \right\} \times \exp\left\{ -\frac{g^2}{2} \int_\Omega \mathrm{d}^4x\, \left(\overline{\psi}_b \gamma^5 \gamma^\mu (\lambda^A)_{bc} \psi_c\right)^2 (x) \right\} \times \exp\left\{ -i \int_\Omega \mathrm{d}^4x\, (\overline{\psi}_a\, \eta_a + \overline{\eta}_a \psi_a)(x) \right\}. \quad (26)$$

To proceed with a bosonization analysis of the fermion field theory described by this path integral, it appears to be convenient to write the interaction Lagrangian in a form closely parallel to the usual fermion-vector coupling in gauge theories by making use of an auxiliary non-abelian vector field $A_\mu^a(x)$, but with a purely imaginary coupling with the axial vectorial fermion current (in the Euclidean world).

$$Z[\eta_a, \overline{\eta}_a] = \frac{1}{Z(0,0)} \int \prod_{a=1}^{N^2-N} D[\psi_a(x)] D\left[\overline{\psi}_a(x)\right] \times \int \prod_{a=1}^{N^2-N} \prod_{\mu=0}^{3} D\left[A_\mu^a(x)\right] \times \exp\Big\{ -\frac{1}{2} \int_\Omega \mathrm{d}^4x (\psi_a, \overline{\psi}_a) \times \begin{bmatrix} 0 & \partial\!\!\!/ + ig\gamma_5\, A\!\!\!/ \\ (\partial\!\!\!/ + ig\gamma_5\, A\!\!\!/)^* & 0 \end{bmatrix} \begin{pmatrix} \psi_a \\ \overline{\psi}_a \end{pmatrix} (x) \Big\} \times \exp\left\{ -\frac{1}{2} \int_\Omega \mathrm{d}^4x (A_\mu^a\, A_\mu^a)(x) \right\} \times \exp\left\{ -i \int_\Omega \mathrm{d}^4x (\overline{\psi}_a\, \eta_a + \overline{\eta}_a\, \psi_a)(x) \right\}. \quad (27)$$

At this point of our analysis we present our idea to bosonize (solve) exactly this fermion path integral. The main point is to use the old suggestion that, for a strong coupling and a large number of colors (the t'Hooft limit), one should expect a great reduction of the (continuum) vector-dynamical degrees of freedom to a manifold of constant-gauge fields living on the infinite-dimensional Lie algebra of $SU(\infty)$ [1,6]. In this t'Hooft limit of a large number of colors, we can evaluate exactly the fermion path integral by noting that the Dirac kinetic operator in the presence of the constant $SU(N)$ gauge fields can be written in the following suitable form

$$\exp\Big\{ -\frac{1}{2} \int_\Omega \mathrm{d}^4x (\psi_a \overline{\psi}_a) \begin{bmatrix} 0 & U(\varphi)\, \partial\!\!\!/ U(\varphi) \\ U(\varphi)^* & \partial\!\!\!/^* U^*(\varphi) \end{bmatrix} \times \begin{pmatrix} \psi_a \\ \overline{\psi}_a \end{pmatrix} (x) \Big\} \quad (28)$$

where the chiral Hermitian phase-factor is given by

$$U(\varphi) = \exp[-g\gamma_5 (A_\mu^a x^\mu) \lambda_a] \quad (29)$$

with the chiral $SU(N)$-valued phase defined by the constant-gauge-field configuration

$$\varphi(x^\mu) = \varphi^a \lambda_a = (A_\mu^a x^\mu) \lambda_a \quad (30)$$

Note that, due to the attractive coupling of the axial current-axial current interaction of our Thirring model (26), the axial vector coupling is made of an imaginary complex coupling constant ig.

Now we can follow exactly as in the well-known chiral path-integral bosonization scheme [5,7] to solve the quark-field path integral (28) exactly by means of the chiral change of variables

$$\psi(x) = \exp\{-g\, \gamma_5\, \varphi(x)\} \chi(x)\,, \quad (31)$$

$$\overline{\psi}(x) = \overline{\chi}(x) \exp\{-g\, \gamma_5\, \varphi(x)\}\,. \quad (32)$$

After implementing the variable change from (31) to (32), the fermion sector of the generating functional takes the form where the independent Euclidean fermion fields are decoupled from the interacting intermediating non-abelian constant vector field A_μ^a, namely

$$Z[\eta_a,\, \overline{\eta}_a] = \frac{1}{Z(0,0)} \int \prod_{a=1}^{N^2-N} D[\chi_a(x)] D[\overline{\chi}_a(x)] \times \int_{-\infty}^{+\infty} \prod_{a=1}^{N^2-N} \mathrm{d}[A_\mu^a] \times \exp\left\{ +\frac{V}{2}\, \mathrm{Tr}_{SU(N)} \left(A_\mu^2\right) \right\} \times \det{}_F^{+1} \left[(\partial\!\!\!/ + ig\gamma_5\, A\!\!\!/)(\partial\!\!\!/ + ig\gamma_5\, A\!\!\!/)^* \right] \times \exp\left\{ -\frac{1}{2} \int_\Omega \mathrm{d}^4x\, (\chi_a, \overline{\chi}_a) \begin{bmatrix} 0 & \partial\!\!\!/ \\ \partial\!\!\!/^* & 0 \end{bmatrix} \begin{pmatrix} \chi_a \\ \overline{\chi}_a \end{pmatrix} (x) \right\} \times \exp\Big\{ -i \int_\Omega \mathrm{d}^4x \left(\overline{\chi}_a e^{-g\gamma_5 \varphi(x)} \eta_a + \overline{\eta}_a e^{-g\gamma_5 \varphi(x)} \chi_a \right) \times (x) \Big\}. \quad (33)$$

Let us now evaluate exactly the fermionic functional determinant of (33), which is given by the functional Jacobian associated with the chiral fermion-field reparameterizations in (31) and (32).

To compute this fermionic determinant, $\ell n \det_F^{+1}[(\partial\!\!\!/ + ig\ A\!\!\!/)(\partial\!\!\!/ + ig\ A\!\!\!/)^*]$, we use the well-known theorem of Schwarz-Romanov [7] by introducing a σ parameter ($0 \leq \sigma \leq 1$) dependent family of interpolating Dirac operators (see (23) in Sect. 2).

$$\begin{aligned} \not{D}^{(\sigma)} &= \left(\not\partial + ig \not{A}^{(\sigma)}\right) \\ &= \exp\{-g\,\sigma\gamma_5\,\varphi(x)\}(\not\partial)\exp\{-g\,\sigma\gamma_5\varphi(x)\}\,. \end{aligned} \tag{34}$$

Since we have the relationship for the interpolating Dirac operators

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\not{D}^{(\sigma)} = (-g\gamma_5\varphi)\,\not{D}^{(\sigma)} + \not{D}^{(\sigma)}(-g\gamma_5\varphi) \tag{35}$$

and the usual proper-time definition for the functional determinants under analysis

$$\begin{aligned} &\log\det{}_F^{+1}\left(\not{D}^{(\sigma)}\,\not{D}^{(\sigma)^*}\right) \\ &= \lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{ds}{s}\,\mathrm{Tr}_F\left(e^{-s(\not{D}^{(\sigma)}\not{D}^{(\sigma)^*}}\right), \end{aligned} \tag{36}$$

one obtains straightforwardly the following differential equation for the fermionic functional determinant

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\sigma}\left\{\log\det{}_F^{+1}\left(\not{D}^{(\sigma)}\,\not{D}^{(\sigma)^*}\right)\right\} \\ &= 4\lim_{\varepsilon\to 0}\left\{\int \mathrm{d}^4x\,\mathrm{Tr}_F\left[g\gamma_5\varphi\times\exp\left(-\varepsilon\,\not{D}^{(\sigma)}\,\not{D}^{(\sigma)^*}\right)\right]\right\} \end{aligned} \tag{37}$$

where Tr_F denotes the complete trace over the color, Dirac and spacetime indices. At this point we note that the diagonal part of $\exp(-\varepsilon\,\not{D}^{(\sigma)}\,\not{D}^{(\sigma)^*})$ has a well-known gauge-invariant asymptotic expansion in four-dimensions [4] (where $\sigma^{\mu\nu} = \frac{1}{2i}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$)

$$\begin{aligned} &\exp\left(-\varepsilon\,\not{D}^{(\sigma)}\,\not{D}^{(\sigma)^*}\right) \\ &= \frac{1}{4\pi^2}\left\{\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}(F^b_{\mu\nu}(\sigma A)\sigma^{\mu\nu}\lambda_b)\right. \\ &\quad + \frac{1}{4}\left(-\frac{1}{3}F^b_{\mu\nu}(\sigma A)F^{b'}_{\mu\nu}(\sigma A)\lambda_b\lambda_{b'}\right. \\ &\quad\quad \left. - \frac{1}{2}F^c_{\alpha\beta}(\sigma\ A)F^{c'}_{\alpha'\beta'}(\sigma A)\lambda_c\lambda_{c'}\gamma^\alpha\gamma^\beta\gamma^{\alpha'}\gamma^{\beta'}\right) \\ &\quad \left. + 0(\varepsilon)\right\}. \end{aligned} \tag{38}$$

After substituting the Seeley-Hadamard expansion into (38), by taking into account (30), together with the fact that $\mathrm{Tr}_{\mathrm{Dirac}}(\gamma_5) = 0$ and $\mathrm{Tr}_{\mathrm{Dirac}}(\gamma_5\,\sigma^{\mu\nu}) = 0$, one obtains finally the only possible non-zero term in our evaluations

$$\begin{aligned} W[A^a_\mu] &= 16\left\{\left(\int_\Omega \mathrm{d}^4x\,\frac{(-g)}{(4\pi)^2}\left(-\frac{1}{8}\right)x^\mu\right)\right. \\ &\quad \times(\sigma A^a_\mu)(F^c_{\alpha\beta}(\sigma A)^*F^{c'}_{\alpha\beta}(\sigma A)\,\mathrm{Tr}_{SU(N)}(\lambda_a\lambda_c\lambda_{c'}))\,. \end{aligned} \tag{39}$$

By supposing explicit spacetime symmetry of the finite-volume region Ω, one has that the symmetry integral vanishes

$$\int_\Omega \mathrm{d}^4x\cdot x^\mu \equiv 0\,. \tag{40}$$

As a consequence, we get the somewhat expected result that the fermion functional determinant in the presence of constant-gauge external fields coincides with the free one, (see (23)) namely:

$$\det{}_F\left[(\not\partial + ig\not{A})(\not\partial + ig\not{A})^*\right]\Big/\det{}_F\left[(\not\partial)(\not\partial)^*\right] = 1\,. \tag{41}$$

Let us return to our bosonized generating functional (after substituting the above results into (33).)

$$\begin{aligned} &Z[\eta_a,\overline{\eta}_a] \\ &= \frac{1}{Z(0,0)}\int\prod_{a=1}^{N^2-N} D[\chi_a(x)]D[\overline{\chi}_a(x)] \\ &\quad\times\int_{-\infty}^{+\infty}\prod_{a=1}^{N^2-N}\mathrm{d}[A^a_\mu]\exp\left\{+\frac{1}{2}V\,\mathrm{Tr}_{SU(N)}(A_\mu)^2\right\} \\ &\quad\times\exp\left\{-\frac{1}{2}\int_\Omega \mathrm{d}^4x(\chi_a,\overline{\chi}_a)\begin{bmatrix}0 & \not\partial\\ \not\partial^* & 0\end{bmatrix}\begin{pmatrix}\chi_a\\ \overline{\chi}_a\end{pmatrix}(x)\right\} \\ &\quad\times\exp\left\{-i\int_\Omega \mathrm{d}^4x\right. \\ &\quad\quad\left.\times\left(\overline{\chi}_a e^{-g\gamma_5(A^a_\mu\lambda_a)x^\mu}\eta_a + \overline{\eta}_a e^{-g\gamma_5(A^a_\mu\lambda_a)x^\mu}\chi_a\right)(x)\right\}. \end{aligned} \tag{42}$$

Let us argue in favor of the theory's triviality by analyzing the long-distance behavior associated to the $SU(N)$ gauge-invariant fermionic composite operator $B(x) = \psi_a(x)\,\overline{\psi}_a(x)$. It is straightforward to obtain its exact expression from the bosonized path integral (42)

$$\begin{aligned} &\Big\langle B(x)\,B(y)\Big\rangle \\ &= \Big\langle(\chi_a(x)\,\overline{\chi}_a(x))(\chi_a(y)\,\overline{\chi}_a(y))\Big\rangle^{(0)}\times G((x-y)) \end{aligned} \tag{43}$$

here the reduced model's gluonic factor is given exactly in its structural analytical form by the path integral (without the γ_5 Dirac indices)

$$\begin{aligned} G((x-y)) &\sim \frac{1}{G(0)}\int_{-\infty}^{+\infty}\prod_{a=1}^{N^2-N}\mathrm{d}[A^a_\mu] \\ &\quad\times\exp\left\{+\frac{1}{2}\,\mathrm{vol}(\Omega)\,\mathrm{Tr}_{SU(N)}(A_\mu)^2\right\} \\ &\quad\times\mathrm{Tr}_{SU(N_c)}\,\mathbb{P}\left\{\exp -g\oint_{C_{xy}} A_\alpha\mathrm{d}x_\alpha\right\} \end{aligned} \tag{44}$$

where C_{xy} is a planar closed contour containing the points x and y and possessing an area S given roughly by the factor $S = (x-y)^2$.

The notation $\langle\ \ \rangle^{(0)}$ means that the fermionic average is defined solely by the fermion free action as given in the decoupled form (42).

Let us pass to the important step of evaluating the Wilson phase-factor average in (44) at the t'Hooft limit of a large number of colors $N\to\infty$. As the first step to implement such evaluation, let us consider our loop C_{xy} as a closed contour lying on the plane $\mu = 0$, $\nu = 1$ bounding the planar region S (see Sect. 2)

We now observe that the ordered phase factor for constant-gauge fields can be exactly evaluated by means of a triangularization of the planar region S, i.e

$$S = \bigcup_{l=1}^{M} \Delta_{\mu\nu}^{(i)} . \tag{45}$$

Here, each counterclockwise-oriented triangle $\triangle_{\mu\nu}^{(i)}$ is adjacent to the next one, and $\triangle_{\mu\nu}^{(i)} \cap \triangle_{\mu\nu}^{(i+1)}$ is the common side with the opposite orientations.

At this point we note that

$$\mathbb{P}\left\{ e^{-g\int_{\Delta_{\mu\nu}^{(i)}} A_\alpha \cdot \mathrm{d}x_\alpha} \right\} \cong e^{-gA_\alpha \cdot \ell_\alpha^{(1)}} \cdot e^{-gA_\alpha \cdot \ell_\alpha^{(2)}} \cdot e^{-gA_\alpha \cdot \ell_\alpha^{(3)}} \tag{46}$$

where $\{\ell_\alpha^{(i)}\}_{1=1,2,3}$ are the triangle sides satisfying the (vector) identity $\ell_\alpha^{(1)} + \ell_\alpha^{(2)} + \ell_\alpha^{(3)} \equiv 0$.

Since we have that

$$\mathbb{P}\left\{ e^{-g\oint_{(x)} A_\alpha \mathrm{d}x_\alpha} \right\} = \lim_{n\to\infty} \prod_{i=1}^{n} \mathbb{P}\left\{ e^{-g\int_{\Delta_{\mu\nu}^{(i)}} A_\alpha \mathrm{d}x_\alpha} \right\} \tag{47}$$

and by using the Campbell-Hausdorff formulae to sum the product limit (47) with X and Y denoting general elements of the $SU(N)$ Lie algebra:

$$e^X \cdot e^Y = e^{X+Y+\frac{1}{2}[X,Y]} + 0(g^2) \tag{48}$$

one arrives at the non-abelian Stokes theorem for constant-gauge fields (see second reference in [1]).

$$\mathbb{P}\left\{ e^{-g\int_{C_{xy}} A_\alpha \mathrm{d}x_\alpha} \right\} = \mathbb{P}\left\{ e^{-g\iint_S F_{01}\mathrm{d}\sigma^{01}} \right\} = \mathbb{P}\left\{ e^{+(g)^2\,[A_0,A_1]\cdot S} \right\} . \tag{49}$$

As a consequence, we have the following result (exact at $N \to \infty$) to be used in our analysis below

$$\mathrm{Tr}_{SU(N)}\, \mathbb{P}\left\{ e^{-g\int_{C_{xy}} A_\alpha dx_\alpha} \right\} \sim \exp\left\{ + \frac{(g^2S)^2}{2} (\mathrm{Tr}_{SU(N)}[A_0, A_1])^2 \right\} + O\left(\frac{1}{N}\right) . \tag{50}$$

Note that (50) is a rigorous result and (49) is a rigorous proof of the non-abelian Stokes theorem as used in Sect. 2.

Let us now substitute (50) into (44) and, taking into account the natural two-dimensional degrees of freedom reduction of the average (44), we have

$$G((x-y)) = \frac{1}{\tilde{G}(0)} \int_{-\infty}^{+\infty} \prod_{a=1}^{N^2-N} \mathrm{d}[A_1^a]\,\mathrm{d}[A_0^a] \times \exp\left\{ +\frac{1}{2} V\left[\mathrm{Tr}_{SU(N)}(A_0^2 + A_1^2) \right] \right. \times \exp\left\{ + \frac{(g^2S)^2}{2} (\mathrm{Tr}_{SU(N)}[A_0, A_1])^2 \right\} \tag{51}$$

where $\tilde{G}(0)$ is the normalization factor given explicitly by

$$\tilde{G}(0) = \int_{-\infty}^{+\infty} \prod_{a=1}^{N^2-N} \mathrm{d}[A_1^a]\mathrm{d}[A_0^a] \times \exp\left\{ -\frac{1}{2}\,\mathrm{vol}(\Omega)[(A_0^a)^2 + (A_1^a)^2] \right\} . \tag{52}$$

By looking closely at (51) and (52), one can see that the behavior of the Wilson phase-factor average at large N is asymptotic to the value of the integral below

$$G((x-y))_{N\gg1} \sim \left\{ \int_{-\infty}^{+\infty} \mathrm{d}a \exp\left\{ -\frac{1}{2}\,\mathrm{vol}(\Omega)a^2 \right\} \exp\left\{ -\frac{(g^2S)^2}{2} a^4 \right\} \times \left(\int_{-\infty}^{+\infty} \mathrm{d}a\, \exp\left\{ -\frac{1}{2}\,\mathrm{vol}(\Omega)a^2 \right\} \right)^{-1} \right\}^{N^2-N} . \tag{53}$$

By using the well-known result (see [9], p. 307, (3).323-3)

$$\int_0^\infty \exp\left(-\beta^2x^4 - 2\gamma^2x^2\right)\mathrm{d}x = 2^{-\frac{3}{2}} \left(\frac{\gamma}{\beta}\right) e^{\frac{\gamma^4}{2\beta^2}} K_{\frac{1}{4}}\left(\frac{\gamma^4}{2\beta^2}\right) \tag{54}$$

we obtain the closed result (at finite volume $V = \mathrm{vol}(\Omega) < \infty$).

$$G((x-y))_{N\gg1} \sim \left\{ \left(\frac{\sqrt{\mathrm{vol}(\Omega)}N}{2\cdot\left(\frac{g^2SN}{\sqrt{2}}\right)} \right) e^{+\frac{(\mathrm{vol}(\Omega))^2}{\frac{32}{N^2}\left(\frac{g^2SN}{\sqrt{2}}\right)^2}} \times K_{\frac{1}{4}}\left(\frac{(\mathrm{vol}(\Omega))^2\,N^2}{16g^4N^2S^2} \right) \times \left(\frac{\sqrt{\pi}}{2\cdot\left(\frac{\mathrm{vol}(\Omega)}{2}\right)^{\frac{1}{2}}} \right)^{-1} \right\}^{N^2-N} . \tag{55}$$

Let us now give a theoretical physicist's argument for the theory's triviality at infinite volume $\mathrm{vol}(\Omega) \to \infty$ on the basis of the explicit representation. Let us firstly define the limit of the infinite-volume theory by means of the following limit

$$\mathrm{vol}(\Omega) = S^2 \tag{56}$$

and consider the asymptotic limit of the correlation function at $|x-y| \to \infty$ $(S \to \infty)$.

By using the standard asymptotic limit of the Bessel function

$$\lim_{z\to\infty} K_{\frac{1}{4}}(z) \sim e^{-z}\sqrt{\frac{\pi}{2z}} \tag{57}$$

one obtains the result ($\lim_{N\to\infty} g^2 N = g_\infty^2 < \infty$) in four dimensions

$$G((x-y))_{\substack{N\gg 1\\ |x-y|\to\infty}}$$
$$\sim \lim_{S\to\infty}\left\{\frac{N}{S}\cdot e^{\frac{N^2S^4}{16S^2}}\sqrt{\frac{16\pi}{N^2S^22}}e^{-\frac{S^2N^2}{16}}\right\}^{N^2-N}$$
$$\sim \frac{1}{|x-y|^{4(N^2-N)}}\,. \tag{58}$$

So, we can see that for N a very large parameter, there is a fast decay of (58) without any bound on the power decay law. However in the usual LSZ framework for quantum fields, it the opposite behavior would be expected through the lack of decay of a factor as in the two-dimensional case (see (58) for vol(Ω) = S), meaning physically that one can observe fermionic scattering free states at large separation. However at $N\to\infty$, where we expect the full validity of our analysis, one obtains [on the basis of the formal behavior of (58)] the vanishing of the fermionic correlation function in (43), faster than any power of $|x-y|$ for large $|x-y|$. This result shows that g^2_{bare} may be zero from the very beginning and strongly signalling the fact that the chiral Thirring model for large number of colors may remain a trivial quantum field theory, a result that is not fully expected in view of previous claims on the subject that large-N resummations always turn non-renormalizable field theories into non-trivial renormalizable useful ones [8]. However, rigorous mathematical proofs are needed to establish such an important triviality result in full [8].

Finally, and as a last remark on our (55–58), let us point out that a mathematical rigorous sense in which to consider these results is by taking as our continuum spacetime Ω, a set formed of n hyper-four-dimensional cubes of side a – the expected size of the non-perturbative vacuum domain of our theory (see the first reference in [1]) – and the surface S being formed, for instance, by n squares on the Ω plane section contained on the plane $\mu = 0$, $\nu = 1$. As a consequence of this construction, we can see that the large behavior is given exactly by

$$G(na)_{N\gg 1} \overset{n\to\infty}{\sim} \left\{\frac{N}{\overline{g}_\infty^2\cdot na^2}\, e^{\frac{(N^2n^2a^8}{32.\left(\frac{\overline{g}_\infty^2 na^2}{\sqrt{2}}\right)^2}}\right.$$
$$\left.\times K_{\frac{1}{4}}\left(\frac{N^2(n^2a^8)}{16(g_\infty^2)^2\,n^2a^4}\right)\right\}^{N^2-N}$$
$$\sim \left(\frac{1}{na^4}\right)^{N^2-N}$$
$$\sim e^{-N(N-1)\ell g(na^4)} \underset{N\to\infty}{\sim} 0\,. \tag{59}$$

4 The loop space argument for the Thirring model triviality

To argue one more time for the triviality phenomenon of the $SU(N)$ non-abelian Thirring model of Sect. 3 for finite N, let us consider the generating functional (27) for vanishing fermionic sources $\eta_a = \overline{\eta}_a = 0$, the so-called vacuum-energy theory's content or the theory's partition functional

$$Z(0,0) = \int \prod_{a=1}^{N^2-N}\prod_{\mu=0}^{3} D[A_\mu^a(x)]\, e^{-\frac{1}{2}\int_\Omega \mathrm{d}^4x(A_\mu^a\,A_\mu^a)(x)}$$
$$\times \det{}_F[(\not\partial + ig\gamma_5\not A)(\not\partial + ig\gamma_5\not A)^*]\,. \tag{60}$$

At this point of our analysis, let us write the functional determinant of (60) as a functional on the space of closed bosonic paths $\{X_\mu(\sigma),\ 0\le\sigma\le T,\ X_\mu(0) = X_\mu(T) = x_\mu\}$, namely [6] and first reference on [8].

$$\ell g \det{}_F[(\not\partial + ig\gamma_5\not A)(\not\partial + ig\gamma_5\not A)^*]$$
$$= \sum_{C_{xx}}\left\{\mathbb{P}_{SU(N)}\cdot\mathbb{P}_{\text{Dirac}}\right.$$
$$\times\exp\left[-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma)\mathrm{d}X_\mu(\sigma)\right.$$
$$\left.\left.+\frac{i}{2}\left[\gamma^\alpha,\gamma^\beta\right]\oint_{C_{xx}} F_{\alpha\beta}(X_\beta(\sigma))\mathrm{d}s\right]\right\}. \tag{61}$$

The sum over the closed loops C_{xy} with fixed end-point x_μ is given by the proper-time bosonic path integral below

$$\sum_{C_{xx}} = -\int_0^\infty \frac{\mathrm{d}T}{T}\int \mathrm{d}^4x_\mu \int_{\chi_\mu(0)=x_\mu=\chi_\mu(T)} D^F[X(\sigma)]$$
$$\times\exp\left\{-\frac{1}{2}\int_0^T \dot{X}^2(\sigma)\mathrm{d}\sigma)\right\}. \tag{62}$$

Note the symbols of the path ordenation $\mathbb{P}$ of both the Dirac and color indices on the loop hase space factors in the expression (61).

By using the Mandelstam area-derivative operator $\delta/\delta\,\sigma_{\gamma\rho}(X(\sigma))$ [4], one can rewrite (61) into the suitable form as an operation in the loop space with Dirac matrices bordering the loop C_{xx}, namely:

$$\ell g \det{}_F[(\not\partial + ig\gamma_5\not A)(\not\partial + ig\gamma_5\not A)^*]$$
$$= \sum_{C_{xx}} \mathbb{P}_{\text{Dirac}}\exp\left\{\oint_{C_{xx}}\mathrm{d}\sigma\frac{i}{2}\left[\gamma^\alpha,\gamma^\beta\right](\sigma)\frac{\delta}{\delta\sigma_{\alpha\beta}(X(\sigma))}\right.$$
$$\left.\times\,\mathbb{P}_{SU(N)}\left[\exp(-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma)\mathrm{d}X_\mu(\sigma))\right]\right\}. \tag{63}$$

To show the triviality of functional fermionic determinant when averaging over the (white-noise) auxiliary non-abelian fields as in (60), we can use a cumulant expansion, which in generic form reads

$$\langle e^f\rangle_{A_\mu} = \exp\left\{\langle f\rangle_{A_\mu} + \frac{1}{2}\left(\langle f^2\rangle_{A_\mu} - \langle f\rangle^2_{A_\mu}\right) + \dots\right\}. \tag{64}$$

So let us evaluate explicitly the first order cumulant

$$\sum_{C_{xy}} \mathbb{P}_{\text{Dirac}}\left\{\oint_{C_{xy}} \mathrm{d}s\,\frac{i}{2}\left[\gamma^\alpha(\sigma),\gamma^\beta(\sigma)\right]\frac{\delta}{\delta\sigma_{\alpha\beta}(X(\sigma))}\right.$$
$$\times\left\langle \mathbb{P}_{SU(N)}\left[\exp(-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma))\mathrm{d}X_\mu(\sigma)\right]\right\rangle_{A_\mu} \tag{65}$$

with the average $\langle\ \ \rangle_{A_\mu}$ defined by the path integral (60).

By using the Grassmanian zero-dimensional representation to write explicitly the $SU(N)$ path order as a Grassmanian path integral [10]

$$\mathbb{P}_{SU(N)}\left[\exp(-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma))dX_\mu(\sigma))\right] \tag{66}$$
$$= \int\prod_{a=1}^{N^2-N} D^F\left[\theta_a(\sigma)\right] D^F[\theta^*_a(\sigma)](\sum_{a=1}^{N^2-N}\theta_a(0)\,\theta^*_a(T))$$
$$\times\exp\left(\frac{i}{2}\int_0^T \mathrm{d}\sigma\right.$$
$$\left.\sum_{a=1}^{N^2-N}\left(\theta_a(\sigma)\frac{\mathbf{d}}{\mathrm{d}\sigma}\theta^*_a(\sigma)+\theta^*_a(\sigma)\frac{\mathrm{d}}{\mathrm{d}\sigma}\theta_a(\sigma)\right)\right)$$
$$\times\exp\left(g\int_0^T \mathrm{d}\sigma(A^a_\mu(X^\beta(\sigma))(\theta_b(\lambda_a)_{bc}\theta^*_c)(\sigma)\mathrm{d}X^\mu(\sigma))\right)$$

one can easily see that the average over the $A_\mu(x)$ fields is straightforward and produces as a result the following self-avoiding loop action

$$\left\langle\mathbb{P}_{SU(N)}\left[\exp(-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma))\mathrm{d}X_\mu(\sigma))\right]\right\rangle_{A_\mu}$$
$$= \int\prod_{a=1}^{N^2-N} D^F[\theta_a(\sigma)]\, D^F[\theta^*_a(\sigma)](\sum_{a=1}^{N^2-N}\theta_a(0)\theta^*_a(T))$$
$$\times\exp\left(\frac{i}{2}\int_0^T\mathrm{d}\sigma\right.$$
$$\left.\times\sum_{a=1}^{N^2-N}\left(\theta_a(\sigma)\frac{\mathbf{d}}{\mathrm{d}\sigma}\theta^*_a(\sigma)+\theta^*_a(\sigma)\frac{\mathbf{d}}{\mathrm{d}\sigma}\theta_a(\sigma)\right)\right)$$
$$\times\exp\left\{\frac{g^2}{2}\int_0^T\mathrm{d}\sigma\int_0^T\mathrm{d}\sigma'\right.$$
$$\times\left[(\theta_b(\lambda_a)_{bc}\theta^*_c)(\sigma)(\theta_b(\lambda_a)_{bc}\theta^*_c)(\sigma')\right]$$
$$\left.\times\,\delta^{(D)}(X_\mu(\sigma)-X_\mu(\sigma'))\mathrm{d}X_\mu(\sigma)\mathrm{d}X_\mu(\sigma')\right\}. \tag{67}$$

At this point one can use the famous probabilistic topological Parisi argument [11] to show the $\lambda\varphi^4$ triviality at the four-dimensional spacetime [8]: due to the fact that the Hausdorff dimension of our Brownian loops $\{X_\mu(\sigma)\}$ is two, and that the topological rule for a continuous manifold holds true in the present situation, one obtains that, for ambient space greater than (or equal) to four, the Hausdorff dimension of the closed path intersection set of the argument of the delta function in (67) is empty. So, we have as a consequence

$$\left\langle\mathbb{P}_{SU(N)}\left[\exp(-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma)\mathrm{d}X_\mu(\sigma))\right]\right\rangle_{A_\mu} = 1\,. \tag{68}$$

Proceeding in an analogous way for higher-order cumulants, one uses again the aforementioned Parisi topological argument to arrive at the general results for a set of m Brownian paths $\{C^{(\ell)}_{xx}\}_{\ell=1,\dots,m}$

$$\left\langle\prod_{\ell=1}^{m}\left[P_{SU(N)}\exp\left(-g\oint_{C^{(\ell)}_{xx}} A_\mu(X^{(\ell)}_\beta(\sigma))\mathrm{d}X^{(\ell)}_\mu(\sigma)\right)\right]\right\rangle_{A_\mu}$$
$$= 1\,. \tag{69}$$

At this point we note that for finite N_c the following result holds true as a consequence of (60) and (69)

$$Z(0,0)$$
$$= \left\langle\exp\left\{\sum_{C_{xx}} P_{\text{Dirac}}\left\{\oint_{C_{xx}}\mathrm{d}\sigma\frac{i}{2}\left[\gamma^\alpha,\gamma^\beta\right](\sigma)\frac{\delta}{\delta\sigma_{\alpha\beta}(X(\sigma))}\right.\right.\right.$$
$$\left.\left.\left.\times P_{SU(N)}\left[\exp(-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma))\mathrm{d}X_\mu(\sigma)\right]\right\}\right\}\right\rangle_{A_\mu}$$
$$= \exp\left\{\sum_{C_{xx}} P_{\text{Dirac}}\left\{\oint_{C_{xx}}\mathrm{d}\sigma\frac{i}{2}\left[\gamma^\alpha,\gamma^\beta\right](\sigma)\frac{\delta}{\delta\sigma_{\alpha\beta}(X(\sigma))}\right.\right.$$
$$\times\left\langle P_{SU(N)}\left[\exp(-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma))\mathrm{d}X_\mu(\sigma)\right]\right\rangle_{A_\mu}$$
$$+\frac{1}{2}\sum_{C^{(1)}_{xx}}\sum_{C^{(2)}_{xx}}\left\{\oint_{C_{xx}}\mathrm{d}\sigma^1\frac{i}{2}\left[\gamma^\alpha,\gamma^\beta\right](\sigma^1)\frac{\delta}{\delta\sigma_{\alpha\beta}(X^1(\sigma^1))}\right.$$
$$\times\oint_{C^{(2)}_{xx}}\frac{i}{2}\left[\gamma^\rho,\gamma^J\right](\sigma^2)\frac{\delta}{\delta\sigma_{\rho J}(X^2(\sigma^2))}$$
$$\times\left\langle P_{SU(N)}\left[\exp(-g\oint_{C^{(1)}_{xx}} A_\mu(X^1_\beta(\sigma^1))\mathrm{d}X^1_\mu(\sigma^1)\right]\right.$$
$$\left.\times P_{SU(N)}\left[\exp(-g\oint_{C^{(2)}_{xx}} A_\mu(X^2_\beta(\sigma^2))\mathrm{d}X^2_\mu(\sigma^2)\right]\right\rangle_{A_\mu}$$
$$\left.+\dots\right\} = \exp(0) = 1 = \det_F(\not{\partial}\,\not{\partial}^*)\,, \tag{70}$$

which in turn leads to the Thirring model's triviality for spacetime R^D with $D \geq 4$.

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